Tractable sampling numbers and best trigonometric m-term approximation in weighted Wiener spaces

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Chemnitz Summer School on Applied Analysis

18.09.2023

UNIVERSITY OF TECHNOLOGY
IN THE EUROPEAN CAPITAL OF CULTURE
CHEMNITZ

Outline

Joint work with...

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Funded by

Europäische Union

Europa fördert Sachsen.
ESF
Europäischer Sozialfonds

Freistaat Sachsen

Diese Maßnahme wird mitfinanziert durch Steuermittel auf der Grundlage des vom Sächsischen Landtag beschlossenen Haushaltes.

Quasi s-Numbers

For $n \in \mathbb{N}_0$, and $X(\Omega), Y$ quasi-Banach function spaces with a continuous linear embedding $T : X \to Y$ the following (quasi) s-Numbers are defined:

- Sampling numbers (linear and non-linear)

$$\varrho_n(X)_Y = \inf_{t_1 \dots t_n \in \Omega} \inf_{R:\mathbb{C}^n \to Y} \sup_{\|f\|_X \leq 1} \left\| f - R\big(f(t_1) \dots f(t_n)\big) \right\|_Y \quad (1)$$

- Gelfand numbers

$$c_n(T : X \to Y) = \inf \left\{ \sup_{f \in B_X \cap M} \|Tf\|_Y : M \subset X \text{ linear subspace with } \operatorname{codim} M < n \right\} \quad (2)$$

- best trigonometric m-term approximation

$$\sigma_n(X)_Y := \sup_{\|f\|_X \leq 1} \inf_{s \in \Sigma_n} \|f - s\|_Y \quad (3)$$

Main result - Tractable s-numbers

The (first) main contribution of this talk are the following tractable results

$$\sigma_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_\infty \le Cn^{-\frac{1}{2}} d\log(n) \tag{4}$$

and

$$c_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \le Cn^{-\frac{1}{2}} d\log(n) \tag{5}$$

with absolute constants C. Note that the right hand side only depends linearly on the dimension d of the space.

Motivation

- Nguyen Nguyen and Sickel recently studied some s-numbers of mixed Wiener classes in [4], however they studied neither Gelfand numbers, sampling numbers nor best m-term approximation
- new results concerning sampling numbers

Proposition 1 ([2, Jahn, Ullrich and Voigtlaender 2023])

Let $n, d \in \mathbb{N}$ then it holds for a quasi-normed function space with continuous embedding into L_∞

$$\varrho_{n \log(n)^3}\left(\mathcal{F}\right)_2 \lesssim \sigma_n\left(\mathcal{F}\right)_\infty + E_{[-n,n]^d}(\mathcal{F})_\infty. \quad (6)$$

Relations between s-numbers

- Gelfand numbers form a lower bound for the non-linear sampling numbers, in particular it holds
$$\varrho_n\left(\mathcal{A}_p^\alpha(\mathbb{T}^d)\right)_2 \gtrsim c_n(id : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to L_2)$$
- Kolmogorov numbers form a lower bound for the linear sampling numbers, in particular it holds
$$\varrho_n^{\text{lin}}\left(\mathcal{A}_p^\alpha(\mathbb{T}^d)\right)_2 \gtrsim d_n(id : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to L_2)$$

In total the Gelfand and sampling numbers give upper and lower bounds for the non-linear sampling numbers.
One important property of s-numbers is, that for two operators S, R it holds

$$s_n(R \circ S) \leq s_n(R)s_1(S) = s_n(R)\|S\|$$

Mixed Wiener spaces

For $\alpha > 0$ and $0 < p < \infty$ we define the mixed Wiener space $\mathcal{A}_p^\alpha(\mathbb{T}^d) \subset L_1(\mathbb{T}^d)$ via its norm

$$\|f\|_{\mathcal{A}_p^\alpha(\mathbb{T}^d)} = \Big(\sum_{\mathbf{k}\in\mathbb{Z}^d} \prod_{i=1}^d (1+|k_i|)^{\alpha p} |\hat{f}(\mathbf{k})|^p\Big)^{\frac{1}{p}}.$$

For $p = 1$ these spaces are the periodic versions of Barron classes. The space $\mathcal{A}_1^0$ is the original Wiener Algebra $\mathcal{A}$. They have a useful embedding into the sequence spaces

$$A_\alpha f = \Big(\prod_{i=1}^d (1+|k_i|)^\alpha \hat{f}(\mathbf{k})\Big)_{\mathbf{k}\in\mathbb{Z}^d}, \quad \|A_\alpha : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to \ell_p(\mathbb{Z}^d)\| = 1.$$

Theorem 2

For $n, d \in \mathbb{N}$, $0 < p \leq 2$ *and* $\alpha > \left(\frac{p-1}{p}\right)_+$ *it holds*

$$c_n(\mathrm{id} : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \asymp n^{-(\alpha+\lambda)} \log(n)^{(d-1)\alpha} \tag{7}$$

where $\lambda = \frac{1}{p} - \frac{1}{2}$.

Idea of proof: Rewrite

$$\mathrm{id} : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to L_2(\mathbb{T}^d)$$

as

$$D_\alpha(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}} = \left(\prod_{i=1}^{d}(1+|k_i|)^{-\alpha} x_{\mathbf{k}}\right)_{\mathbf{k}\in\mathbb{Z}}.$$

Diagonal operator

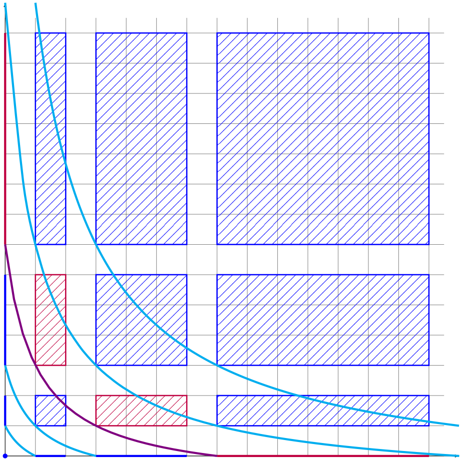
$$\begin{array}{ccc} \mathcal{A}_p^\alpha(\mathbb{T}^d) & \xrightarrow{\mathrm{id}} & L_2(\mathbb{T}^d) \\ \Big\downarrow A_\alpha & & \Big\uparrow B \\ \ell_p(\mathbb{Z}^d) & \xrightarrow[D_\alpha]{} & \ell_2(\mathbb{Z}^d) \end{array}$$

where

$$B(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}} = \frac{1}{\sqrt{2\pi}^d} \sum_{\mathbf{k}\in\mathbb{Z}} x_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}, \quad \|B\| = 1$$

$$c_n(\mathrm{id} : \mathcal{A}_p^\alpha(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = c_n(D_\alpha : \ell_p(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)). \tag{8}$$

Hyperbolic cross



A hyperbolic cross is a set of the form $\left\{\mathbf{n} \in \mathbb{N}_0^d \middle| \prod_{j=1}^d (n_j + 1) \leq c\right\}$. Decompose now $\mathbb{N}_0^d$ in dyadic blocks, where blocks on the same hyperbolic layer have

- the same number of points
- the same maximal weight

Now use this decomposition on D_α. The number of points per layer can now be computed as

$$C_j := \#\square_j = 2^j \binom{j+d-1}{j} \asymp 2^j j^{d-1}.$$

Best m-term approximation of $\mathcal{A}$

Lemma 3

Let $2 \leq q < \infty$ *and* $\alpha > 0$ *then it holds*

$$\sigma_n(\mathcal{A})_q \leq C \frac{q}{\log(q)} n^{-\frac{1}{2}} \tag{9}$$

for an absolute constant $C \geq 1$.

The proof of this Lemma is based on a version of the Rosenthal inequality from probability theory. We can even employ the Nikolskij inequality to get a version of this for $q = \infty$.

Lemma 4

For $N \in \mathbb{N}$ *and a trigonometric polynomial* $t \in \mathcal{T}([-N, N]^d)$ *it holds*

$$\sigma_n(t)_\infty \leq C d \log(N) n^{-\frac{1}{2}} \|t\|_{\mathcal{A}}. \tag{10}$$

Best m-term approximation of $\mathcal{A}_p^\alpha$

Theorem 5

For $n, d \in \mathbb{N}$ *with* $0 < p \leq q$ *and* $2 \leq q \leq \infty$ *as well as* $\alpha > \left(\frac{p-1}{p}\right)_+$ *it holds*

$$n^{-(\alpha+\lambda)} \log(n)^{(d-1)\alpha} \lesssim \sigma_n\left(\mathcal{A}_p^\alpha(\mathbb{T}^d)\right)_q \lesssim n^{-(\alpha+\lambda)} \log(n)^{(d-1)\alpha+\mu} \tag{11}$$

where

$$\lambda = \frac{1}{p} - \frac{1}{2},$$

and $\mu = 1$ *if both* $q = \infty$ *and* $d > 1$ *otherwise* $\mu = 0$.

Linear sampling numbers

Proposition 6 (see [4, Nguyen, Nguyen and Sickel, 2022])

For the Kolmogorov numbers d_n it holds for $\alpha > 0$,

$$d_n\big(id : \mathcal{A}_1^{\alpha}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\big) \asymp n^{-\alpha} \log(n)^{\alpha(d-1)}. \tag{12}$$

Since the Kolmogorov numbers form a lower bound for the linear sampling numbers this immediately gives the following result

$$\varrho_n^{\text{lin}}\big(\mathcal{A}_1^{\alpha}(\mathbb{T}^d)\big)_2 \gtrsim n^{-\alpha} \log(n)^{\alpha(d-1)}. \tag{13}$$

Non-linear sampling numbers

For the non-linear sampling numbers an analogous bound holds in terms of the Gelfand numbers

$$\varrho_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \gtrsim n^{-(\alpha+\frac{1}{2})}\log(n)^{\alpha(d-1)}. \tag{14}$$

Proposition 1 together with Theorem 5 now yields

$$\varrho_n\big(\mathcal{A}_1^\alpha(\mathbb{T}^d)\big)_2 \lesssim n^{-(\alpha+\frac{1}{2})}\log(n)^{\alpha(d-1)+3(\alpha+\frac{1}{2})+1} \tag{15}$$

There is a difference of $\frac{1}{2}$ in the main rate of the decay between the linear and non-linear sampling numbers in mixed Wiener classes measured in L_2.

Tractable bound on the best m-term approximation

Again the original Theorem 5 states

$$\sigma_n\left(\mathcal{A}_1^\alpha(\mathbb{T}^d)\right)_\infty \lesssim n^{-(\alpha+\frac{1}{2})} \log(n)^{(d-1)\alpha+1}$$

Where another 2^d term is hidden by the $\lesssim$. This is not a suitable bound in a setting where $n = d^s$.

Theorem 7

Let $m, d \in \mathbb{N}$ and $\alpha > 0$ then it holds

$$\sigma_n\left(\mathcal{A}_1^\alpha(\mathbb{T}^d)\right)_\infty \leq C n^{-\frac{1}{2}} d \log(n) \tag{16}$$

with absolute constant $C \geq 1$.

This bound decays for $n \geq d^{2+\varepsilon}$, with $\varepsilon > 0$.

Thank you for your attention

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